

WEIGHTED ANISOTROPIC MORREY SPACES ESTIMATES FOR ANISOTROPIC MAXIMAL OPERATORS

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ABSTRACT. The aim of this paper can give weighted anisotropic Morrey Spaces estimates for anisotropic maximal functions.

1. INTRODUCTION

The classical Morrey spaces have been introduced by Morrey in [8] to study the local behavior of solutions of second order elliptic partial differential equations(PDEs). In recent years there has been an explosion of interest in the study of the boundedness of operators on Morrey-type spaces. It has been obtained that many properties of solutions to PDEs are concerned with the boundedness of some operators on Morrey-type spaces. In fact, better inclusion between Morrey and Hölder spaces allows to obtain higher regularity of the solutions to different elliptic and parabolic boundary problems. Chiarenza and Frasca [3] have obtained the boundedness of Hardy-Littlewood maximal operator, the fractional integral operator and a singular integral operator in the Morrey spaces. The boundedness of fractional integral operator has been originally studied by Adams [1]. On the other hand, it is very important to study weighted estimates for these operators in harmonic analysis. On the weighted L_p spaces, the boundedness of operators above has been obtained by Muckenhoupt [9] and Coifman and Fefferman [4]. These results are extended to several spaces, however, weighted Morrey spaces have yet to be studied.

Thus, in this paper we shall introduce the weighted anisotropic Morrey Spaces and investigate the boundedness of the anisotropic maximal functions on this space.

2. DEFINITIONS AND NOTATION

Throughout this paper all notation is standard or will be defined as needed.

Let \mathbb{R}^n be the n -dimensional Euclidean space of points $x = (x_1, \dots, x_n)$ with norm $|x| = \left(\sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}}$, $Q = Q(x_0, r)$ denotes the cube centered at x_0 with side length r . Given a cube Q and $\lambda > 0$, λQ denotes the cube with the same center as Q whose side length is λ times that of Q . A weight is a locally integrable function on \mathbb{R}^n which takes values in $(0, \infty)$ almost everywhere. For a weight function w and a measurable set E , we define $w(E) = \int_E w(x)dx$, the Lebesgue measure of E

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by $|E|$ and the characteristic function of E by χ_E . Given a weight function w , we say that w satisfies the doubling condition if there exists a constant $D > 0$ such that for any cube Q , we have $w(2Q) \leq Dw(Q)$. When w satisfies this condition, we denote $w \in \Delta_2$, for short.

If w is a weight function, we denote by $L_p(w) \equiv L_p(\mathbb{R}^n, w)$ the weighted Lebesgue space defined by the norm

$$\|f\|_{L_{p,w}} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} < \infty, \text{ when } 1 \leq p < \infty$$

and by

$$\|f\|_{L_{\infty,w}} = \operatorname{esssup}_{x \in \mathbb{R}^n} |f(x)| w(x), \text{ when } p = \infty.$$

We denote by $WL_p(w)$ the weighted weak space consisting of all measurable functions f such that

$$\|f\|_{WL_p(w)} = \sup_{t>0} t w(\{x \in \mathbb{R}^n : |f(x)| > t\})^{\frac{1}{p}} < \infty.$$

Let $\mathbb{R}_0^n = \mathbb{R}^n \setminus \{0\}$ and \mathbb{Z} be the set of integer numbers. Let also $a = (a_1, \dots, a_n)$ be a fixed vector from \mathbb{R}^n with $a_i > 0$, $i = 1, \dots, n$. Consider a real $n \times n$ matrix A with eigenvalues λ_j , $\operatorname{Re} \lambda_j = a_j > 0$ and let $Q = \operatorname{tr} A$ be its trace. The matrix A determines a one-parameter group $A_t = \exp(A \ln t)$, $t > 0$ of nonsingular transformations of \mathbb{R}^n . Denote by $\operatorname{diag} \{a_1, \dots, a_n\}$ the matrix with numbers a_1, \dots, a_n on the main diagonal and zero off-diagonal elements and let $a_{\max} = \max_{1 \leq i \leq n} a_i$. Associated with the group A_t is the A_t -homogeneous metric $\rho : \mathbb{R}_0^n \rightarrow \mathbb{R}_+$, $\rho(A_t x) = t \rho(x)$ which is smooth on \mathbb{R}_0^n .

For $x \in \mathbb{R}_0^n$, let $[x]_a$ be a positive solution to the equation $\sum_{i=1}^n x_i^2 [x]_a^{-2a_i} = 1$ and $|x|_a = \max_{1 \leq i \leq n} |x_i|^{\frac{1}{a_i}}$. Note that $\rho(x)$ is equivalent to $|x|_a$, i.e.,

$$c_1 |x|_a \leq \rho(x) \leq c_2 |x|_a.$$

For $x \in \mathbb{R}^n$ and $r > 0$ we define the one-parametric parallelepiped

$$\begin{aligned} E(x, t) &= \{y \in \mathbb{R}^n : |x - y|_a \leq t\} \\ &= \{y \in \mathbb{R}^n : |y_i - x_i| \leq t^{a_i}, i = 1, \dots, n\} \end{aligned}$$

and by $E = E(\alpha)$ we denote the set of all $E(x, t)$ with $x \in \mathbb{R}^n$, $t > 0$. If $a_1 = \dots = a_n$, then $E(x, t)$ is a cube.

All parallelepipeds are assumed to have their sides parallel to the coordinate axes. $E = E(x_0, r)$ denotes the parallelepiped centered at x_0 with side length $r^{\alpha_1}, \dots, r^{\alpha_n}$ consequently. Given a parallelepiped E and $\lambda > 0$, $\lambda^a E$ denotes the parallelepiped with the same center as E whose side length is $(\lambda r)^{a_1}, \dots, (\lambda r)^{a_n}$ consequently.

The letter C is used for various constants, and may change from one occurrence to another. First we introduce a weighted anisotropic Morrey space.

Definition 1. (*Weighted anisotropic Morrey spaces*) Let $1 \leq p < \infty$, $0 \leq \kappa < 1$ and w be a weight. Then a weighted anisotropic Morrey space is defined by

$$L_{p,\kappa,a}(w) := \left\{ f \in L^{loc}(w) : \|f\|_{L_{p,\kappa,a}(w)} < \infty \right\},$$

where

$$\|f\|_{L_{p,\kappa,a}(w)} = \sup_E \left(\frac{1}{w(E)^\kappa} \int_E |f(x)|^p w(x) dx \right)^{\frac{1}{p}}$$

and the supremum is taken over all parallelepipeds E on \mathbb{R}^n . In the case of $a = (1, \dots, 1)$, we get weighted Morrey spaces $L_{p,\kappa}(w) = L_{p,\kappa,1}(w)$.

Remark 1. Alternatively, we could define the weighted Morrey spaces with anisotropic balls instead of parallelepipeds. Hence we shall use these two definitions of weighted anisotropic Morrey spaces appropriate to calculation. Also, we could define the weighted Morrey spaces with cubes instead of parallelepipeds.

Remark 2. (1) If $w \equiv 1$ and $\kappa = \lambda/n$ with $0 < \lambda < n$, then $L_{p,\kappa}(w) = L_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey spaces. If $w \equiv 1$ and $\kappa = \frac{\lambda}{|a|}$ with $0 < \lambda < |a|$, $|a| = a_1 + \dots + a_n$, then $L_{p,\kappa,a}(w) = L_{p,\lambda,a}(w)$ the anisotropic Morrey spaces.

(2) Let $w \in \Delta_2$. If $\kappa = 0$, then $L_{p,0}(w) = L_p(w)$ is the weighted Lebesgue spaces. If $\kappa = 1$, then $L_{p,1}(w) = L_\infty(w)$ by the Lebesgue differentiation theorem with respect to w (see [10]).

(3) In the one-dimensional case, let a weight $w(x) = |x|^\alpha$ for some $-\frac{1}{2} < \alpha < 0$ and a function $f(x) = \chi_{(0,1)} |x|^{-\frac{1}{2}}$. Then $f \in L^{1, \frac{\alpha+1}{\alpha+1}}(w) \setminus L^{2(\alpha+1)}(w)$.

Let $f \in L^{loc}(\mathbb{R}^n)$. The anisotropic maximal function Mf and the sharp maximal function $f^\#$ are defined by

$$Mf(x) = \sup_{t>0} |E(x,t)|^{-1} \int_{E(x,t)} |f(y)| dy$$

and

$$f^\#(x) = \sup_{t>0} |E(x,t)|^{-1} \int_{E(x,t)} |f(y) - f_{E(x,t)}| dy,$$

where $f_{E(x,t)} = |E(x,t)|^{-1} \int_{E(x,t)} |f(y)| dy$.

If $a_1 = \dots = a_n$, then $E(x,t)$ is a cube and Mf becomes the usual Hardy-Littlewood maximal function. For $r > 0$, we denote $M_r f(x)$ by $(M|f|^r(x))^{\frac{1}{r}}$.

Let w be a weight. M_w denotes the anisotropic maximal operator with respect to the measure $w(x) dx$ defined by

$$(2.1) \quad M_w f(x) = \sup_E \frac{1}{w(E)} \int_E |f(y)| w(y) dy$$

We shall end this section by defining two weight classes.

Definition 2. (Muckenhoupt classes) A weight function w is in the Muckenhoupt's class $A_p(\mathbb{R}^n)$ with $1 < p < \infty$, if there exist $C > 1$ such that for any parallelepiped E

$$(2.2) \quad [w]_{A_p(E)} \equiv \left(|E|^{-1} \int_E w(x) dx \right) \left(|E|^{-1} \int_E w(x)^{1-p'} dx \right)^{p-1} \leq C$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and the infimum of C satisfying the following inequality (2.2) is denoted by $[w]_{A_p}$, and also for $p = \infty$ we define $A_\infty = \bigcup_{1 \leq p < \infty} A_p$, $[w]_{A_\infty} = \inf_{1 \leq p < \infty} [w]_{A_p}$ and $[w]_{A_\infty} \leq [w]_{A_p}$.

When $p = 1$, $w \in A_1$ if there exist $C > 1$ such that for almost every x ,

$$(2.3) \quad Mw(x) \leq Cw(x)$$

and the infimum of C satisfying the inequality (2.3) is denoted by $[w]_{A_1}$.

It is easy to verify that, $\rho(x)^\alpha \in A_p$ if and only if $-|a| < \alpha < |a|(p-1)$ for $1 < p < \infty$ and $\rho(x)^\alpha \in A_1$ if and only if $-|a| < \alpha \leq 0$.

3. LEMMAS AND WELL KNOWN RESULTS

In this section, we shall prove some lemmas and describe the well-known result about the weighted L_p spaces.

Theorem 1. ([6], Theorem 2.6, p. 146]) *If $1 < p < \infty$ and $w \in \Delta_2$, then the operator M_w is bounded on $L_p(w)$.*

The next lemma plays an important role in our proofs of theorems. We say that w satisfies the reverse doubling condition if w has the property (3.1) of the following lemma.

Lemma 1. *If $w \in \Delta_2$, then there exists a constant $D_1 > 1$ such that*

$$(3.1) \quad w(2E) \geq D_1 w(E).$$

Proof. Fix a parallelepiped $E = E(x_0, r)$. Then we can choose a parallelepiped $R \subset 2E$ with side length $\frac{r}{2}$ which is disjoint from the parallelepiped E . Hence

$$w(E) + w(R) \leq w(2E).$$

On the other hand, since $E \subset 5R$ we have $w(E) \leq w(5R) \leq D^3 w(R)$, where D is a doubling constant. Therefore we have

$$w(E) + \frac{w(E)}{D^3} \leq w(2E).$$

□

Lemma 2. ([7]) *The following statements hold:*

(1) *If $w \in A_p$ for some $1 \leq p < \infty$, then $w \in \Delta_2$. Moreover, for all $\lambda > 1$ we have*

$$w(\lambda E) \leq \lambda^{np} [w]_{A_p} w(E).$$

(2) *Let $w \in A_p$ for some $1 \leq p < \infty$. Then we have*

$$Mf(x) \leq [w]_{A_p}^{\frac{1}{p}} (M_w(|f|^p)(x))^{\frac{1}{p}}.$$

Proof. (1) Let $w \in A_p$ for some $1 \leq p < \infty$ and $\lambda > 1$. Then

$$\begin{aligned} \frac{w(\lambda E)}{w(E)} &= \left(\frac{|\lambda E|}{|E|} \right)^p \frac{[w]_{A_p}(\lambda E)}{[w]_{A_p}(E)} \frac{\left(\int_E w(x)^{1-p'} dx \right)^{p-1}}{\left(\int_{\lambda E} w(x)^{1-p'} dx \right)^{p-1}} \\ &\leq \lambda^{np} [w]_{A_p}. \end{aligned}$$

(2) Let $w \in A_p$ for some $1 \leq p < \infty$. Applying the Hölder's inequality, we get

$$\begin{aligned} Mf(x) &= \sup_E \frac{1}{|E|} \int_E |f(x)| dx \\ &\leq \sup_E \frac{1}{|E|} \left(\int_E |f(x)|^p w(x) dx \right)^{\frac{1}{p}} \left(\int_E w(x)^{1-p'} dx \right)^{\frac{1}{p'}} \\ &= \sup_E \left(\frac{1}{w(E)} \int_E |f(x)|^p w(x) dx \right)^{\frac{1}{p}} \left(\int_E w(x)^{1-p'} dx \right)^{\frac{1}{p'}} \left(\frac{w(E)}{|E|} \right)^{\frac{1}{p}} \\ &= \left(\sup_E \frac{1}{w(E)} \int_E |f(x)|^p w(x) dx \left(\frac{1}{|E|} \int_E w(x) dx \right) \left(\int_E w(x)^{1-p'} dx \right)^{p-1} \right)^{\frac{1}{p}} \\ &\leq [w]_{A_p}^{\frac{1}{p}} (M_w(|f|^p)(x))^{\frac{1}{p}}. \end{aligned}$$

□

4. ANISOTROPIC MAXIMAL FUNCTION

In this section, we shall state the boundedness of the anisotropic maximal operators on weighted anisotropic Morrey Spaces.

Theorem 2. (Our main result) *If $1 < p < \infty$, $0 < \kappa < 1$ and $w \in \Delta_2$, then the operator M_w is bounded on $L_{p,\kappa,a}(w)$.*

Proof. Fix a parallelepiped $E \subset \mathbb{R}^n$. We decompose $f = f_1 + f_2$, where $f_1 = f\chi_{3E}$. Since M_w is a sublinear operator, we have

$$\begin{aligned} &\left(\int_E M_w f(x)^p w(x) dx \right)^{\frac{1}{p}} \\ &\leq \left(\int_E M_w f_1(x)^p w(x) dx \right)^{\frac{1}{p}} + \left(\int_E M_w f_2(x)^p w(x) dx \right)^{\frac{1}{p}} \\ &= I + II. \end{aligned}$$

For the term I , since M_w is bounded on $L_p(w)$ (see Theorem 1), we obtain

$$\begin{aligned} I &\leq \left(\int_E M_w f_1(x)^p w(x) dx \right)^{\frac{1}{p}} \\ &\leq C \left(\int_{3E} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} \\ &\leq C \|f\|_{L_{p,\kappa,a}(w)} w(E)^{\frac{\kappa}{p}}. \end{aligned}$$

Next we estimate the term II . By simple geometric observation, we have for any $x \in E$, note that for all \tilde{E} such that $x \in \tilde{E}$, $\tilde{E} \cap (3E)^c \neq \emptyset$ there exists R such that $E \subset 3R$ and $\frac{1}{3}R \subset \tilde{E} \subset R$.

Then $w(\tilde{E}) \geq w(\frac{1}{3}R) \geq \frac{1}{D_1} w(R)$ and

$$M_w f_2(x) \leq D_1 \sup_{R: E \subset 3R} \frac{1}{w(R)} \int_R |f(y)| w(y) dy.$$

Note that

$$\begin{aligned} &\frac{1}{w(R)} \int_R |f(y)| w(y) dy \\ &\leq \frac{1}{w(R)} \left(\int_R |f(y)|^p w(y) dy \right)^{\frac{1}{p}} \left(\int_R w(y) dy \right)^{\frac{1}{p'}} \\ &\leq \frac{1}{w(R)^{1-\frac{1}{p'}}} \left(\int_R |f(y)|^p w(y) dy \right)^{\frac{1}{p}} \\ &= \frac{1}{w(R)^{\frac{1}{p}}} \left(\frac{1}{w(R)^\kappa} \int_R |f(y)|^p w(y) dy \right)^{\frac{1}{p}} w(R)^{\frac{\kappa}{p}} \\ &\leq \left(\frac{1}{w(R)^\kappa} \int_R |f(y)|^p w(y) dy \right)^{\frac{1}{p}} w(R)^{\frac{\kappa-1}{p}} \\ &\leq C \|f\|_{L_{p,\kappa,a}(w)} w(R)^{\frac{\kappa-1}{p}}, \end{aligned}$$

if $E \subset 3R$. So we obtain

$$\begin{aligned}
II &= \left(\int_E M_w f_2(x)^p w(x) dx \right)^{\frac{1}{p}} \\
&\leq \left(\int_E \left[\sup_{x \in R} \frac{1}{w(R)} \int_R |f(y)| w(y) dy \right]^p w(x) dx \right)^{\frac{1}{p}} \\
&\leq \left(\int_E \left[c \|f\|_{L_{p,\kappa,a}(w)} w(E)^{\frac{\kappa-1}{p}} \right]^p w(x) dx \right)^{\frac{1}{p}} \\
&\leq c \|f\|_{L_{p,\kappa,a}(w)} w(E)^{\frac{\kappa-1}{p}} \left(\int_E w(x) dx \right)^{\frac{1}{p}} \\
&= c \|f\|_{L_{p,\kappa,a}(w)} w(E)^{\frac{\kappa}{p}}.
\end{aligned}$$

Therefore

$$II \leq C \|f\|_{L_{p,\kappa,a}(w)} w(E)^{\frac{\kappa}{p}}.$$

This completes the proof. \square

Theorem 3. (Our main result) If $1 < p < \infty$, $0 < \kappa < 1$ and $w \in A_p$, then the anisotropic maximal operator M is bounded on $L_{p,\kappa,a}(w)$.

If $p = 1$, $0 < \kappa < 1$ and $w \in A_1$, for all $t > 0$ and any parallelepiped E ,

$$w(x \in E : Mf(x) > t) \leq \frac{C}{t} \|f\|_{L_{1,\kappa,a}(w)} w(E)^\kappa.$$

Proof. Let $1 < p < \infty$. By the reverse Hölder inequality (see [4]), there exists $1 < r < p$ such that $w \in A_r$. Hence it follows from Lemma 2 (2) and Theorem 2 that

$$\begin{aligned}
&\left(\frac{1}{w(E)^\kappa} \int_E Mf(x)^p w(x) dx \right)^{\frac{1}{p}} \\
&\leq C \left(\frac{1}{w(E)^\kappa} \int_E M_w(|f|^r)(x)^{\frac{p}{r}} w(x) dx \right)^{\frac{1}{p}} \\
&\leq C \|M_w(|f|^r)\|_{L_{\frac{p}{r},\kappa,a}(w)}^{\frac{1}{r}} \\
&\leq C \| |f|^r \|_{L_{\frac{p}{r},\kappa,a}(w)}^{\frac{1}{r}} \\
&= c \left[\sup_E \left(\frac{1}{w(E)^\kappa} \int_E |f(x)|^{r(\frac{p}{r})} w(x) dx \right)^{\frac{r}{p}} \right]^{\frac{1}{r}} \\
&= C \|f\|_{L_{p,\kappa,a}(w)}.
\end{aligned}$$

When $p = 1$, we use the Fefferman-Stein maximal inequality

$$\int_{\{x: Mf(x) > t\}} \varphi(x) dx \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)| (M_\varphi)(x) dx$$

for any functions f and $\varphi \geq 0$ (see [5, 6]).

Fix a parallelepiped $E = E(x_0, r)$. Put $\varphi(x) = w(x) \chi_E(x)$. Then we have

$$\begin{aligned} & \int_{\{x: Mf(x) > t\}} \chi_E(x) w(x) dx \\ & \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)| M(w \chi_E)(x) dx \\ & = \frac{C}{t} \left(\int_{3E} + \int_{(3E)^c} \right) = \frac{C}{t} \{I + II\} \end{aligned}$$

for all $t > 0$.

We now estimate the term I . Since $w \in A_1$, it follows that

$$M(w \chi_E)(x) \leq M(w)(x) \leq Cw(x).$$

So it follows that

$$I \leq Cw(3E)^\kappa \|f\|_{L_{1,\kappa,a}(w)} \leq Cw(E)^\kappa \|f\|_{L_{1,\kappa,a}(w)}.$$

To estimate the term II , we consider the form

$$\frac{1}{|R|} \int_{R \cap E} w(y) dy$$

for $x \in (3E)^c \cap R$ and $R \cap E \neq \emptyset$. By simple geometric observation, we have

$$\frac{1}{|R|} \int_{R \cap E} w(y) dy \leq C_n \left(\frac{1}{|x - x_0|_a^{|a|}} \int_E w(y) dy \right) \leq C_n |x - x_0|_a^{-|a|} w(E).$$

Therefore we obtain

$$M(w \chi_E)(x) \leq C_n |x - x_0|_a^{-|a|} w(E).$$

Since $w \in A_1$, we have $w \in \Delta_2$ by Lemma 2 (1). Using Lemma 1, we have $w(3E) \geq w(2E) \geq D_1 w(E)$ with $D_1 > 1$. Thus we can estimate the term II as

follows:

$$\begin{aligned}
II &\leq Cw(E) \int_{(3E)^c} \frac{|f(x)|}{|x-x_0|_a^{|a|}} dx \\
&= C \sum_{j=1}^{\infty} \int_{3^{j+1}E \setminus (3^jE)} \frac{|f(x)|}{|x-x_0|_a^{|a|}} dx \\
&\leq Cw(E) \sum_{j=1}^{\infty} \frac{1}{|3^jE|} \int_{3^{j+1}E} |f(x)| dx \\
&\leq Cw(E) \sum_{j=1}^{\infty} \frac{1}{|3^jE|} \left(\operatorname{esssup}_{x \in 3^{j+1}E} \frac{1}{w(x)} \right) \int_{3^{j+1}E} |f(x)| w(x) dx \\
&= Cw(E) \sum_{j=1}^{\infty} \frac{1}{|3^jE|} \frac{|3^{j+1}E|}{w(3^{j+1}E)} \int_{3^{j+1}E} |f(x)| w(x) dx \\
&= Cw(E)^{\kappa} \sum_{j=1}^{\infty} \frac{w(E)^{1-\kappa}}{w(3^{j+1}E)^{1-\kappa}} \frac{1}{w(3^{j+1}E)^{\kappa}} \int_{3^{j+1}E} |f(x)| w(x) dx \\
&\leq Cw(E)^{\kappa} \|f\|_{L_{1,\kappa,a}(w)} \sum_{j=1}^{\infty} \frac{w(E)^{1-\kappa}}{w(3^{j+1}E)^{1-\kappa}} \\
&\leq Cw(E)^{\kappa} \|f\|_{L_{1,\kappa,a}(w)}.
\end{aligned}$$

The last series converges since the reverse doubling constant is larger than one (see Lemma 1). This completes the proof. \square

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